BEMA 56A Graph Theory

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2.1. The adjacency and incidence of matrices

2.2. Operations on Graphs

2.3. Intersection graphs

2.4 Line Graphs

Definition 1.

Let $G = (V, E)$ be a (p, q) graph.

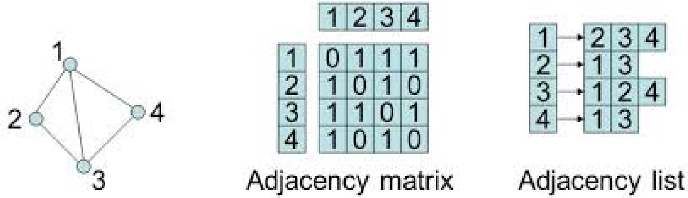
Denote the vertices of G by $v_1, v_2, \ldots, v_p$.

The *adjacency matrix* of G is the matrix $A(G) = (a_{ij})$,

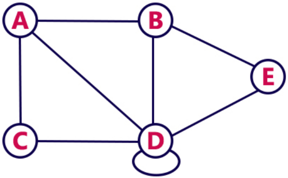
is a $p \times p$ matrix, in which

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Example 1

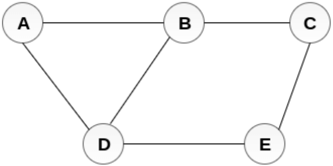

1
2
3
4
1 2 3 4
1 0 1 1 1
2 1 0 1 0
3 1 1 0 1
4 1 0 1 0
Adjacency matrix
1 → 2 3 4
2 → 1 3
3 → 1 2 4
4 → 1 3
Adjacency list

Example 2



	A	B	C	D	E
A	0	1	1	1	0
B	1	0	0	1	1
C	1	0	0	1	0
D	1	1	1	1	1
E	0	1	0	1	0

Example 3



Undirected Graph

	A	B	C	D	E
A	0	1	0	1	0
B	1	0	1	1	0
C	0	1	0	0	1
D	1	1	0	0	1
E	0	0	1	1	0

Adjacency Matrix

Properties

1. For a simple graph G, $a_{ij} \in \{0, 1\}$, and hence $A(G)$ is a $(0, 1)$ matrix.

2. For $i \neq j$, v_i is adjacent to $v_j \Leftrightarrow v_j$ is adjacent to v_i,

   i.e., $a_{ij} = a_{ji}$.

   Hence $A(G) = A(G)^T$. ($A(G)^T$ - is the transpose of $A(G)$.)

   Thus $A(G)$ is symmetry.

3. The sum of the i^{th} row of A is equal to the degree of v_i

Definition 2.

Let $G = (V, E)$ be a (p, q) graph.

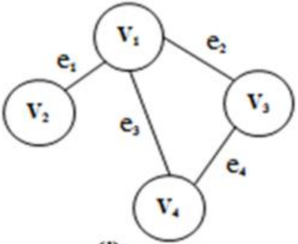
Denote $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots, e_q\}$.

The *incidence matrix* of G is the matrix $B(G) = (b_{ij})$,

is a $p \times q$ matrix, in which

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident with } e_j \\ 0, & \text{otherwise} \end{cases}$$

Example 1

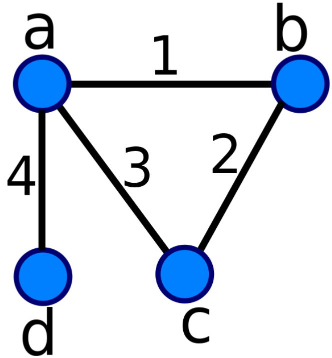


(i)

	e_1	e_2	e_3	e_4
V_1	1	1	1	0
V_2	1	0	0	0
V_3	0	1	0	1
V_4	0	0	1	1

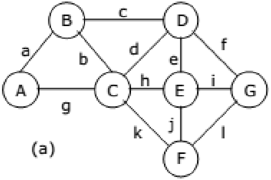
(ii)

Example 2



	1	2	3	4
a	1	0	1	1
b	1	1	0	0
c	0	1	1	0
d	0	0	0	1

Example 3



(a)

(b)

	a	b	c	d	e	f	g	h	i	j	k	l
A	1	0	0	0	0	0	1	0	0	0	0	0
B	1	1	1	0	0	0	0	0	0	0	0	0
C	0	1	0	1	0	0	1	1	0	0	1	0
D	0	0	1	1	1	1	0	0	0	0	0	0
E	0	0	0	0	1	0	0	1	1	1	0	0
F	0	0	0	0	0	0	0	0	0	1	1	1
G	0	0	0	0	0	1	0	0	1	0	0	1

Figure 6.5.4 Graph and its incidence matrix

Properties

1. For a simple graph G, $m_{ij} \in \{0, 1\}$, and hence $B(G)$ is a $(0, 1)$ matrix.
2. The sum of the i^{th} row of A is equal to the degree of v_i.
3. Each column sum is 2.

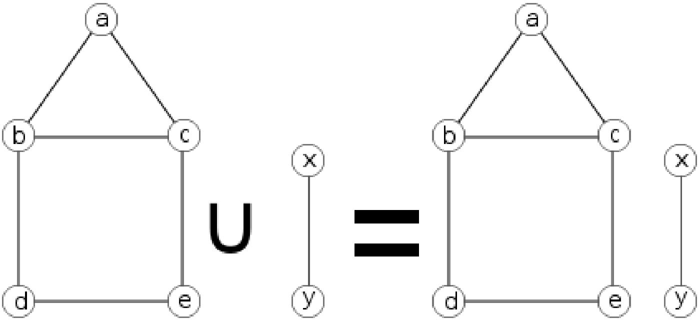
   (since each edge contribute 2 to the degree sum)

Definition 3.

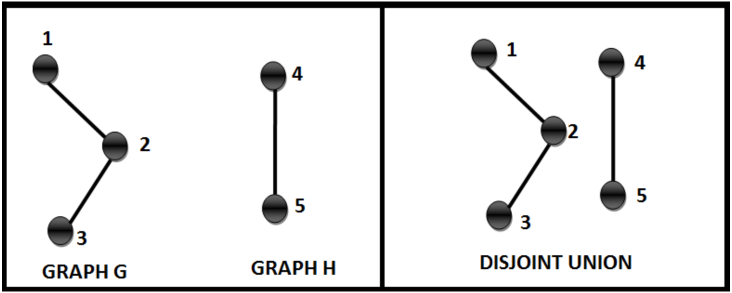
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graph with $V_1 \cap V_2 = \phi$.

Define the union $G_1 \cup G_2$ by $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

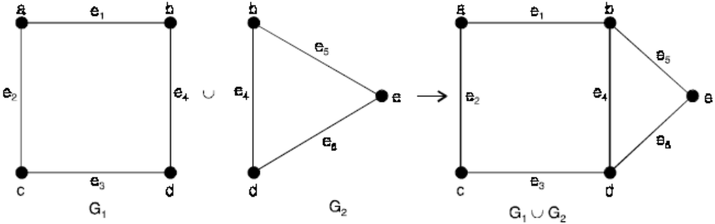
Example 1



Example 2



Example 3



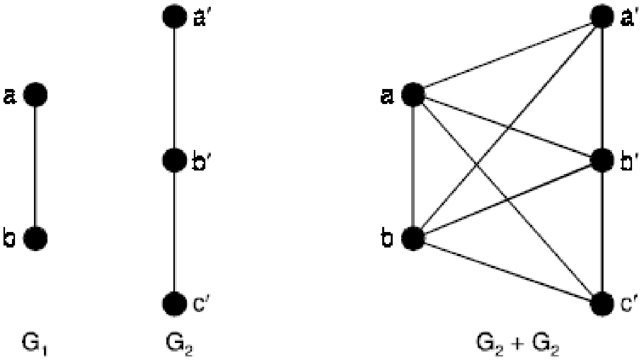
Definition 4.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graph with $V_1 \cap V_2 = \phi$.

Define the sum $G_1 + G_2$ by $G_1 \cup G_2$ together with

all the edges joining vertices of V_1 to vertices of V_2.

Example 1



Definition 5.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graph with $V_1 \cap V_2 = \phi$.

Define the product $G_1 \times G_2$ by its vertex set $V = V_1 \times V_2$ and

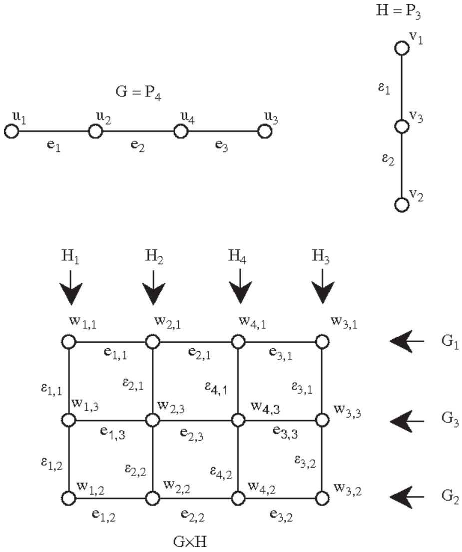
two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$

$$\Longleftrightarrow$$

$u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or

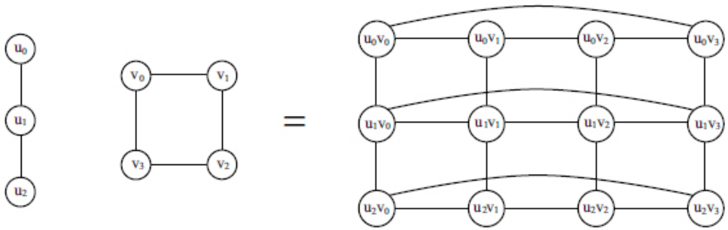
u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$.

Example 1

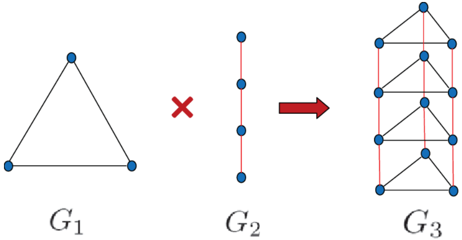


H = P3
v1
ε1
v3
ε2
v2
G = P4
u1
u2
u4
u3
e1
e2
e3
H1
H2
H4
H3
w1,1
w2,1
w4,1
w3,1
G1
e1,1
e2,1
e3,1
ε1,1
ε2,1
ε4,1
ε3,1
w1,3
w2,3
w4,3
w3,3
G3
e1,3
e2,3
e3,3
ε1,2
ε2,2
ε4,2
ε3,2
w1,2
w2,2
w4,2
w3,2
G2
e1,2
e2,2
e3,2
G×H

Example 2


u_0
u_1
u_2
v_0
v_1
v_3
v_2
=
u_0v_0
u_0v_1
u_0v_2
u_0v_3
u_1v_0
u_1v_1
u_1v_2
u_1v_3
u_2v_0
u_2v_1
u_2v_2
u_2v_3

Example 3



Definition 6.

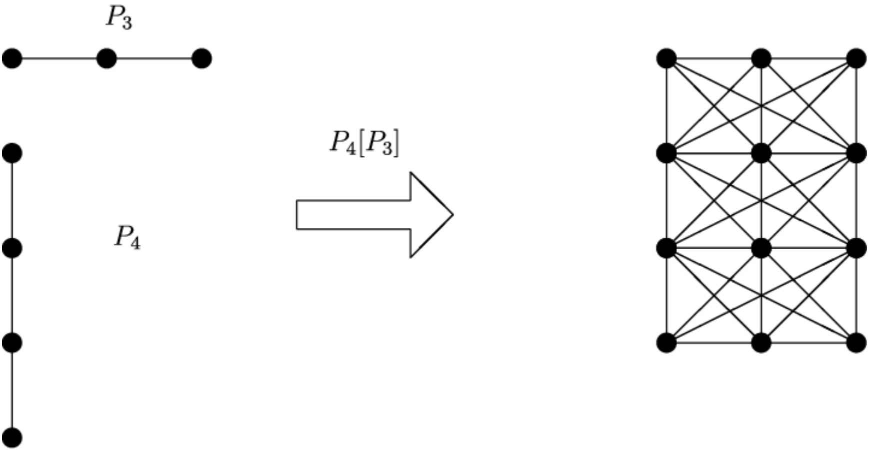
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graph with $V_1 \cap V_2 = \phi$.

Define the composition $G_1[G_2]$ by its vertex set $V = V_1 \times V_2$ and two

vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1\,[G_2]$ if, and only if,

u_1 adjacent to v_1 in G_1 or $u_1 = v_1$ and u_2 adjacent to v_2 in G_2.

Example 1



Theorem 2.1.

Let $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ be two graphs. Then

(i) $G_1 \cup G_2$ is a $(p_1 + p_2, q_1 + q_2)$ graph

(ii) $G_1 + G_2$ is a $(p_1 + p_2, q_1 + q_2 + p_1 p_2)$ graph

(iii) $G_1 \times G_2$ is a $(p_1 p_2, q_1 p_2 + q_2 p_1)$ graph

(iv) $G_1[G_2]$ is a $(p_1 p_2, p_1 q_2 + {p_2}^2 q_1)$ graph

Proof.

Given $G_1 = (p_1, q_1)$ and $G_2 = (p_2, q_2)$ be the two graphs.

(i) $G_1 \cup G_2$:

By def, the number of vertices in $G_1 \cup G_2$ is

$= $ the number of vertices in $G_1 +$ the number of vertices in G_2

$= p_1 + p_2.$

The number of edges in $G_1 \cup G_2$ is

$= $ the number of edges in $G_1 +$ the number of edges in G_2

$= q_1 + q_2.$

Thus, $G_1 \cup G_2$ is a $(p_1 + p_2, q_1 + q_2)$ graph.

(ii) $G_1 + G_2$:

By def, the number of vertices in $G_1 + G_2$ is

$= $ the number of vertices in $G_1 +$ the number of vertices in G_2

The number of edges in $G_1 + G_2$ is

$=$ the number of edges in $G_1 +$ the number of edges in G_2

$+$ the number of edges joining vertices of G_1 to vertices of G_2

$= q_1 + q_2 + p_1 p_2$

Thus, $G_1 + G_2$ is a $(p_1 + p_2, q_1 + q_2 + p_1 p_2)$ graph.

(iii) $G_1 \times G_2$:

By def, the number of vertices in $G_1 \times G_2$ is

$= $ the number of vertices in $G_1 \times$ the number of vertices in G_2

$= p_1 \, p_2$

By def, two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$

$\Leftrightarrow$ either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or

u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$.

$\Rightarrow$ $deg\,(u_i, v_j) = deg\,(u_i) + deg\,(v_j)$ for every i and j.

The total number of edges in $G_1 \times G_2$ is

$$= \frac{1}{2}\left(\sum_{i,j} deg\,(u_i) + deg\,(v_j)\right)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} (\,deg\,(u_i) + deg\,(v_j)\,)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1}\left(\sum_{j=1}^{p_2} deg\,(u_i) + \sum_{j=1}^{p_2} deg\,(v_j)\right)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1} (\, p_2\, deg\,(u_i) \,+\, 2q_2\,)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{p_1} \, p_2\, deg\,(u_i) \,+\, \sum_{i=1}^{p_1} \, 2q_2\right]$$

$$= \frac{1}{2}\,[\, p_2\, 2q_1 \,+\, p_1\, 2q_2\,]$$

$$= \, p_2\, q_1 \,+\, p_1\, q_2$$

Thus, $G_1 \times G_2$ is a $(p_1 p_2, q_1 p_2 + q_2 p_1)$ graph.

(iv) $G_1\,[G_2]$:

By def, the number of vertices in $G_1\,[G_2]$ is

$=$ the number of vertices in $G_1 \times$ the number of vertices in G_2

$= \; p_1\, p_2$

By def, two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1\,[G_2]$

$\Leftrightarrow$ u_1 is adjacent to v_1 in G_1 or either $u_1 = v_1$ and

u_2 is adjacent to v_2 in G_2

$\Rightarrow$ $deg\,(u_i, u_j) = p_2 \deg\,(u_i) + deg\,(u_j)$ for every i and j.

The total number of edges in $G_1\,[G_2]$ is

$$= \frac{1}{2} \left(\sum_{i,j} p_2\, deg\,(u_i) + deg\,(v_j) \right)$$

$$= \frac{1}{2} \left[\sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (\, p_2\, deg\,(u_i) + deg\,(v_j)\,) \right]$$

$$= \frac{1}{2} \left[\sum_{i=1}^{p_1} \left(\sum_{j=1}^{p_2} p_2\, deg\,(u_i) + \sum_{j=1}^{p_2} deg\,(v_j) \right) \right]$$

$$= \frac{1}{2} \left[\sum_{i=1}^{p_1} \left(p_2^2 \, deg\,(u_i) \; + \; 2q_2 \right) \right]$$

$$= \frac{1}{2} \left(\sum_{i=1}^{p_1} \; p_2^2 \, deg\,(u_i) \; + \; \sum_{i=1}^{p_1} \; 2q_2 \right)$$

$$= \frac{1}{2} \left(p_2^2 \, 2q_1 \; + \; p_1 \, 2q_2 \right)$$

$$= \; p_2^2 \, q_1 \; + \; p_1 \, q_2$$

Thus, $G_1 \, [G_2]$ is a $(p_1 p_2 \, , \; q_1 p_2^2 \; + \; q_2 p_1)$ graph.

Definition 7.

Let S be a non-empty set.

Let $\mathcal{F} = \{S_1, S_2, \ldots, S_p\}$ be a family of distinct non-empty subsets of S.

The **intersection** graph of $\mathcal{F}$ is denoted by $\Omega(\mathcal{F})$.

The vertex set of $\Omega(\mathcal{F})$ is $\mathcal{F}$ itself.

Two vertices S_i, S_j $(i \neq j)$ are adjacent if $S_i \cap S_j \neq \phi$

Definition 8.

A graph G is called an **intersection graph** on S if there exists a family $\mathcal{F}$ of subsets of S such that G is isomorphic to $\Omega(\mathcal{F})$

Theorem 2.2.

Every graph is an intersection graph.

Proof:

Let $G = (V, E)$ graph.

Let $V = \{v_1, v_2, v_3, \ldots, v_p\}$.

Take $S = V \cup E$.

For each $v_i \in V$.

Let $S_i = \{v_i\} \cup \{e \in E \mid v_i \in e\}$.

Clearly, $\mathcal{F} = \{S_1, S_2, S_3, \ldots, S_p\}$ is a family of distinct

non-empty subsets of S.

Further v_i, v_j are adjacent in G then $v_i v_j \in S_i \cap S_j$

$$\implies S_i \cap S_j \neq \phi.$$

Conversely, suppose $\mathcal{F} = \{S_1, S_2, S_3, \ldots, S_p\}$ is a family of distinct

non-empty subsets of S.

If $S_i \cap S_j \neq \phi$.

Then there is an element common to $S_i \cap S_j$ is the edge joining v_i to v_j.

So that v_i, v_j are adjacent in G.

$\therefore f : V \longrightarrow \mathcal{F}$ is defined by $f(v_i) = S_i$.

Clearly, f is an isomorphism of G to $\Omega(\mathcal{F})$.

Hence G is an intersection graph.

Definition 9.

Let $G = (V, E)$ be a graph with $E \neq \phi$.

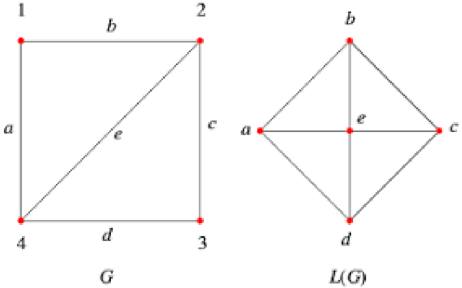
Then the line graph of G is denoted by $L(G)$.

The vertices of $L(G)$ are the edges of G and

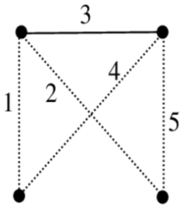
two vertices in $L(G)$ are adjacent if and only if the corresponding

edges are adjacent in G.

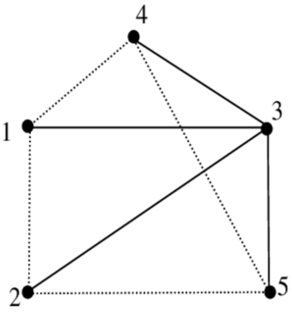
Example 1



Example 2

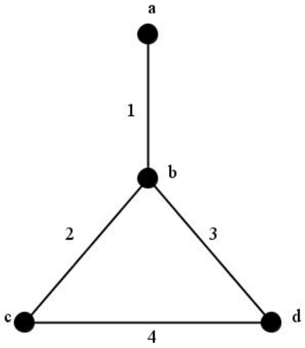


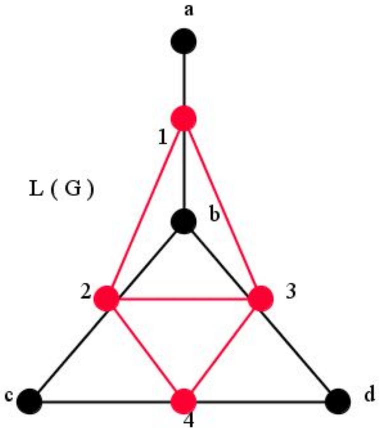
G



L(G)

Draw the line graph $L(G)$ for the following.




a
1
L (G)
b
2
3
c
4
d

Theorem 2.3.

Let $G = (p, q)$ graph.

Then $L(G)$ is a (q, q_L) graph where

$$q_L = \frac{1}{2}\left(\sum_{i=1}^{p} {d_i}^2\right) - q.$$

Proof.

By def, the number of vertices in $L(G)$ is q.

Choose $v_i \in V(G)$.

Then $d(v_i) = d_i$. (say)

i.e., d_i edges incident with v_i in G.

i.e., these d_i edges are adjacent in $L(G)$.

Hence we get $\dfrac{d_i(d_i-1)}{2}$ edges in $L(G)$.

Hence

$$q_L = \sum_{i=1}^{p} \frac{d_i(d_i-1)}{2}$$

$$q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}\left(\sum_{i=1}^{p} d_i\right)$$

By Euler's theorem,

$$\sum_{i=1}^{p} d_i = 2q$$

$$\therefore q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - \frac{1}{2}(2q)$$

$$q_L = \frac{1}{2}\left(\sum_{i=1}^{p} d_i^2\right) - q$$